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Midpoint Integral Inequalities of Hermite-Hadamard Type Using Generalized Fractional Integrals for P-convex and Quasi-convex Stochastic Process

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Abstract

This paper introduces a new definition that generalizes significant fractional integral operators and applies them to P-convex and quasi-convex stochastic processes, resulting in new Hermite-Hadamard type inequalities. These inequalities yield specific results for the integral operators and enhance the understanding of the relationships between convex functions, fractional integral operators, and stochastic processes. These findings provide a foundation for further research, potentially uncovering new mathematical relationships and applications in related areas.

Keywords: stochastic process; quasi convex; P-convex; fractional integrals; fractional operators; Hermite Hadamard inequality.

1 Introduction and Preliminary

1.1 Introduction

Stochastic processes have recently seen a renewed surge of interest across diverse fields. Key contributions include Xiu's introduction of generalized polynomial chaos expansions in [15] for uncertainty quantification, Karatzas and Shreve's foundational work on stochastic calculus and its applications in finance [4] and Skowronski's significant extension of convex function properties to stochastic processes in [14] leading to several novel discoveries. Subsequent research, including the exploration of convex stochastic processes in [7] and investigations into second-sense stochastic processes in [13] has further advanced our understanding of these processes. A comprehensive overview of fundamental stochastic processes, encompassing stochastic calculus (including Lévy processes), Markov models and semi-Markov models along with various examples and applications can be found in [10]. Furthermore, the authors in [8] demonstrate the application of stochastic processes in the context of random walks, highlighting their significance in addressing challenges associated with non-normality and boundedness.

Inequalities play a fundamental role across various mathematical disciplines, finding extensive applications in fields such as physics and engineering. Their evolution, as explored in [2], highlights their transformation from isolated instances to a well-established and rigorous area of study. The Hermite-Hadamard inequality with its significant geometric interpretation and diverse applications exemplifies this evolution. As a cornerstone of convex analysis, the Hermite-Hadamard inequality remains highly significant. Indeed, the paramount importance of mathematical inequalities across a wide array of scientific disciplines cannot be overstated as emphasized in [11].

This paper aims to extend existing fractional integral operators by utilizing a fractional integral operator within its definition. This novel approach enables us to establish Hermite-Hadamard type inequalities for stochastic processes. Specifically, we focus on estimating the left-hand side of these inequalities for stochastic processes whose first derivatives exhibit P-convexity or Quasi-convexity in absolute value. Through this investigation, we aim to deepen our understanding of the properties and behavior of stochastic processes.

1.2 Preliminaries

In the context of probability theory, let $(\mathcal{E}, \mathcal{T}, \mathcal{P})$ denote a probability space, where $\mathcal{E}$ represents the sample space, $\mathcal{T}$ is the sigma-algebra of measurable sets, and $\mathcal{P}$ is the probability measure. If the sigma-algebra $\mathcal{T}$ is measurable then the function $\Re_v, \mathcal{E} \to \mathbb{R}$ is considered a random variable.

A function $\mathcal{S}_p, \mathcal{I} \times \mathcal{E} \to \mathbb{R}$, within an interval $\mathcal{I} \subset \mathbb{R}$, is regarded as a stochastic process if it is a random variable, for every $\chi \in \mathcal{I}$, $\mathcal{S}_p(\chi, \cdot)$ A function $\mathcal{S}_p, \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is considered a stochastic process within an interval $\mathcal{I} \subset \mathbb{R}$ if it satisfies the following properties;

1. The function $\mathcal{S}_p$ is a random variable for every $\chi \in \mathcal{I}$. This means that for each fixed χ, the function $\mathcal{S}_p$ maps each element ω in the sample space $\mathcal{E}$ to a real number, and this mapping is measurable with respect to the sigma-algebra $\mathcal{T}$.

2. The stochastic process $\mathcal{S}_p$ exhibits dependence on both the parameter χ and the elements of the sample space $\mathcal{E}$. This dependence allows for the modeling of random phenomena that evolve over time or space.

By satisfying these properties, the function $\mathcal{S}_p$ effectively captures the randomness and dynamics inherent in a stochastic process, positioning it as a valuable tool in probability theory and various fields of study.

In our previous work [11], we introduced the concepts of mean square continuity (MS-C), mean-square differentiability (MS-D), and mean-square integrability (MS-I). Readers are encouraged to refer to [11] for a detailed discussion of these fundamental concepts and their significance within the context of this study.

We now proceed to define P-convexity and Quasi-convexity for stochastic processes.

Definition 1.1. [11] *A stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is defined as:*

1. *P-convex if for all $\tau \in [0;1]$ and $\mu, \nu \in \mathcal{I}$, the following inequality holds,*
$$\mathcal{S}_p(\tau\mu + (1-\tau)\nu, \cdot) \leq \mathcal{S}_p(\mu, \cdot) + \mathcal{S}_p(\nu, \cdot). \tag{1}$$
2. *Quasi-convex if for all $\tau \in [0;1]$ and $\mu, \nu \in \mathcal{I}$, we have*
$$\mathcal{S}_p(\tau\mu + (1-\tau)\nu, \cdot) \leq \operatorname{Max} \mathcal{S}_p(\mu, \cdot); \mathcal{S}_p(\nu, \cdot). \tag{2}$$

Theorem 1.1. [7] *For any $(\gamma, \delta) \in \mathcal{I}^2$, a stochastic process $\mathcal{S}_p, \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ that is both Jensen-convex and MS-C on $\mathcal{I}$, satisfies the following,*
$$\mathcal{S}_p\left(\frac{\gamma+\delta}{2}, \cdot\right) \leq \frac{1}{\delta-\gamma}\int_\gamma^\delta \mathcal{S}_p(\chi, \cdot)d\chi \leq \frac{\mathcal{S}_p(\gamma, \cdot) + \mathcal{S}_p(\delta, \cdot)}{2}. \tag{3}$$

In this section we introduce a new integral definitions that will prove useful in establishing the main theorems.

Definition 1.2. [11] *In this section we introduce a new integral definitions that will prove useful in establishing the main theorems. $\zeta, [0, \infty) \to [0, \infty)$ is a function that verifies the conditions below,*
$$\int_0^1 \frac{\zeta(\tau)}{\tau} d\tau < \infty, \tag{4}$$
$$\frac{1}{\Upsilon_1} \leq \frac{\zeta(\gamma)}{\zeta(\delta)} \leq \Upsilon_1, \qquad \text{for} \quad \frac{1}{2} \leq \frac{\gamma}{\delta} \leq 2, \tag{5}$$
$$\frac{\zeta(\delta)}{\delta^2} \leq \Upsilon_2 \frac{\zeta(\gamma)}{\gamma^2}, \qquad \text{for} \quad \gamma \leq \delta, \tag{6}$$
$$\left|\frac{\zeta(\delta)}{\delta^2} - \frac{\zeta(\gamma)}{\gamma^2}\right| \leq \Upsilon_3 |\delta-\gamma| \frac{\zeta(\delta)}{\delta^2}, \qquad \text{for} \quad \frac{1}{2} \leq \frac{\gamma}{\delta} \leq 2, \tag{7}$$
with Υ_1, Υ_2, and Υ_3 are positive and independent of γ and δ, where γ and δ are also positive. Suppose $\zeta(\delta)\delta^\alpha$ is an increasing function for some $\alpha \geq 0$, and $\dfrac{\zeta(\delta)}{\delta^\beta}$ is a decreasing function for some $\beta \geq 0$. In this case, according to [12], ζ satisfies (9)–(12).

Based on this, we define the left and right-sided generalized fractional integral operators for a stochastic process $\mathcal{S}_p$ as follows,
$${}_{\gamma^+}I_\zeta \mathcal{S}_p(\chi, \cdot) = \int_\gamma^\chi \frac{\zeta(\chi-\tau)}{(\chi-\tau)} \mathcal{S}_p(\tau, \cdot) d\tau, \quad \text{for} \quad \chi > \gamma, \tag{8}$$
$${}_{\delta^-}I_\zeta \mathcal{S}_p(\chi, \cdot) = \int_\chi^\delta \frac{\zeta(\tau-\chi)}{(\tau-\chi)} \mathcal{S}_p(\tau, \cdot) d\tau, \quad \text{for} \quad \chi < \delta. \tag{9}$$

Generalized fractional integrals provide numerous advantages by encompassing various types of other fractional integrals. These include the Riemann-Liouville fractional integral,conformable fractional integral, k-Riemann-Liouville fractional integral, and others. Notably, (8) and (9) represent these integral operators, serving as a framework that encompasses and extends these aforementioned special cases. This framework allows for a more comprehensive understanding and analysis of fractional integration techniques.

Remark 1.1.

i) For $\zeta(\tau)=\tau$, the operators (8) and (9) transform in to the Riemann integral as follows,

$$I^{\alpha}_{\gamma+}\mathcal{S}_p(\chi)=\int_{\gamma}^{\chi}\mathcal{S}_p(\tau,\cdot)d\tau,\quad \chi>\gamma, \tag{10}$$

$$I^{\alpha}_{\delta-}\mathcal{S}_p(\chi)=\int_{\chi}^{\delta}\mathcal{S}_p(\tau,\cdot)d\tau,\quad \chi<\delta. \tag{11}$$

ii) For $\zeta(\tau)=\dfrac{\tau^{\alpha}}{\xi(\alpha)}$, the operators (8) and (9) become the Riemann Liouville fractional integral,

$$I^{\alpha}_{\gamma+}\mathcal{S}_p(\chi)=\frac{1}{\xi(\alpha)}\int_{\gamma}^{\chi}(\chi-\tau)^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau,\quad \chi>\gamma, \tag{12}$$

$$I^{\alpha}_{\delta-}\mathcal{S}_p(\chi)=\frac{1}{\xi(\alpha)}\int_{\chi}^{\delta}(\tau-\chi)^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau,\quad \chi<\delta. \tag{13}$$

Where $\xi(\alpha)=\int_0^{\infty}e^{-\chi}u^{\alpha-1}\,\mathrm{d}\chi$, moreover it is worth noting that when $\alpha=1$, the expression simplifies to the classical integral.

iii) For $\zeta(\tau)=\dfrac{1}{k\mathcal{I}_k(\alpha)}\tau^{\frac{\alpha}{k}}$, the operators (8) and (9) transform in to the k-Riemann-Liouville fractional integral,

$$I^{\alpha}_{\gamma+,k}\mathcal{S}_p(\chi)=\frac{1}{k\xi_k(\alpha)}\int_{\gamma}^{\chi}(\chi-\tau)^{\frac{\alpha}{k}-1}\mathcal{S}_p(\tau,\cdot)d\tau,\quad \chi>\gamma, \tag{14}$$

$$I^{\alpha}_{\delta-,k}\mathcal{S}_p(\chi)=\frac{1}{k\xi_k(\alpha)}\int_{\chi}^{\delta}(\tau-\chi)^{\frac{\alpha}{k}-1}\mathcal{S}_p(\tau,\cdot)d\tau,\quad \chi<\delta, \tag{15}$$

where

$$\mathcal{I}_k(\alpha)=\int_0^{\infty}\tau^{\alpha-1}e^{-\frac{\tau^k}{k}}d\tau,\quad \mathcal{R}(\alpha)>0, \tag{16}$$

and

$$\mathcal{I}_k(\alpha)=k^{\frac{\alpha}{k}-1}\xi\left(\frac{\alpha}{k}\right),\quad \mathcal{R}(\alpha)>0; k>0, \tag{17}$$

are given by Mubeen and Habibullah [9].

iv) For $\zeta(\tau)=\tau(\chi-\tau)^{\alpha-1}$, we obtain the conformable fractional operators from the operator (13),

$$I^{\alpha}_{\gamma}\mathcal{S}_p(\chi)=\int_{\gamma}^{\chi}\tau^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau,\quad \gamma<\chi=\int_{\gamma}^{\chi}\mathcal{S}_p(\tau,\cdot)d_{\alpha}\tau,\quad \chi>\gamma, \tag{18}$$

is given by Khalil and Horani [5].

v) For $\zeta(\tau) = \dfrac{\tau}{\alpha}\exp\left(-\frac{(1-\alpha)}{\alpha}\tau\right)$*, the exponetial fractional integral operators appear from the operators* (8) *and* (9)*, for* $\alpha \in (0,1)$*,*

$$\mathcal{I}^{\alpha}_{\gamma+}\mathcal{S}_p(\chi,\cdot) = \frac{1}{\alpha}\int_{\gamma}^{\chi}\exp\left(-\frac{(1-\alpha)}{\alpha}(\chi-\tau)\right)\mathcal{S}_p(\tau,\cdot)d\tau, \qquad \gamma < \chi, \tag{19}$$

$$\mathcal{I}^{\alpha}_{\delta-}\mathcal{S}_p(\chi,\cdot) = \frac{1}{\alpha}\int_{\chi}^{\delta}\exp\left(-\frac{(1-\alpha)}{\alpha}(\tau-\chi)\right)\mathcal{S}_p(\tau,\cdot)d\tau, \qquad \chi < \delta, \tag{20}$$

are defined by Kirane and Torebek [1].

2 Midpoint Inequalities for Generalized Fractional Integrals for Stochastic Process

During the course of the study, we establish the following,

$$\Psi(\chi) = \int_0^{\chi}\frac{\zeta((\delta-\gamma)u)}{u}du < \infty, \tag{21}$$

and

$$\phi(\chi) = \int_{\chi}^{1}\frac{\zeta((\delta-\gamma)u)}{\gamma}du < \infty. \tag{22}$$

See [1].

Lemma 2.1. *Consider an MS-D stochastic process* $\mathcal{S}_p : \mathcal{I}\times\mathcal{E} \to \mathbb{R}$*, and* γ *and* δ *are elements of* $\mathcal{I}^{\circ}$*, with* $\gamma \leq \delta$*.*

Suppose $\mathcal{S}'_p$ *is an MS-I on the interval* $[\gamma,\delta]$*. In this case, the following equality for generalized fractional integrals holds,*

$$\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{1}{2\Psi(1)}\left[{}_{\gamma+}I_{\zeta}\mathcal{S}_p(\delta,\cdot) + {}_{\delta-}I_{\zeta}\mathcal{S}_p(\gamma,\cdot)\right] = \frac{\delta-\gamma}{2\Psi(1)}\sum_{k=1}^{4}\mathcal{J}k, \tag{23}$$

where

$$\mathcal{J}1 = \int_0^{\frac{1}{2}}\Psi(\tau)\mathcal{S}'_p(\tau\delta+(1-\tau)\gamma,\cdot)d\tau, \qquad \mathcal{J}2 = \int_0^{\frac{1}{2}}(-\Psi(\tau))\mathcal{S}'_p(\tau\gamma+(1-\tau)\delta,\cdot)d\tau,$$

$$\mathcal{J}3 = \int_{\frac{1}{2}}^{1}(-\Phi(\tau))\mathcal{S}'_p(\tau\delta+(1-\tau)\gamma,\cdot)d\tau, \qquad \mathcal{J}4 = \int_{\frac{1}{2}}^{1}\Phi(\tau)\mathcal{S}'_p\left(\tau\gamma+(1-\tau)\delta,\cdot\right)d\tau.$$

Proof. By calculating $\mathcal{J}1$, $\mathcal{J}2$, $\mathcal{J}3$ and $\mathcal{J}4$, we get,

$$\mathcal{J}1 = \frac{1}{(\delta-\gamma)}\int_0^{\frac{1}{2}}\frac{\zeta((\delta-\gamma)u)}{u}du\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{1}{(\delta-\gamma)}\int_0^{\frac{1}{2}}\frac{\zeta((\delta-\gamma)\tau)}{\tau}\mathcal{S}_p(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,$$

$$\mathcal{J}2 = \frac{1}{(\delta-\gamma)}\int_0^{\frac{1}{2}}\frac{\zeta((\delta-\gamma)u)}{u}du\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{1}{(\delta-\gamma)}\int_0^{\frac{1}{2}}\frac{\zeta((\delta-\gamma)\tau)}{\tau}\mathcal{S}_p(\tau\gamma+(1-\tau)\delta,\cdot)d\tau,$$

$$\mathcal{J}3 = \frac{1}{(\delta-\gamma)}\int_{\frac{1}{2}}^{1}\frac{\zeta((\delta-\gamma)u)}{u}du\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{1}{(\delta-\gamma)}\int_{\frac{1}{2}}^{1}\frac{\zeta((\delta-\gamma)\tau)}{\tau}\mathcal{S}_p(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,$$

$$\mathcal{J}4 = \frac{1}{(\delta-\gamma)}\int_{\frac{1}{2}}^{1}\frac{\zeta((\delta-\gamma)u)}{u}du\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{1}{(\delta-\gamma)}\int_{\frac{1}{2}}^{1}\frac{\zeta((\delta-\gamma)\tau)}{\tau}\mathcal{S}_p(\tau\gamma+(1-\tau)\delta,\cdot)d\tau.$$

And so,

$$\sum_{k=1}^{4}\mathcal{J}k=\frac{2}{(\delta-\gamma)}\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)\int_0^1\frac{\zeta((\delta-\gamma)u)}{u}du$$
$$-\frac{1}{\delta-\gamma}\left(\int_0^1\frac{\zeta((\delta-\gamma)\tau)}{\tau}\mathcal{S}_p(\tau\delta+(1-\tau)\gamma,\cdot)d\tau+\int_0^1\frac{\zeta((\delta-\gamma)\tau)}{\tau}\mathcal{S}_p(\tau\gamma+(1-\tau)\delta,\cdot)d\tau\right)$$
$$=\frac{2}{\delta-\gamma}\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{1}{\delta-\gamma}\Big[_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\Big].$$

We deduce the result by dividing by $2\Psi(1)$ and multiplying by $(\delta-\gamma)$. □

Remark 2.1.

- *For* $\zeta(\tau)=\tau$, (21) *becomes,*

$$\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{1}{(\delta-\gamma)}\int_\gamma^\delta\mathcal{S}_p(\tau,\cdot)\,d\tau$$
$$=(\delta-\gamma)\int_0^{\frac{1}{2}}\tau\Big(\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)-\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)\Big)d\tau.$$

A similar result for functions can be found in [6].

- *For* $\zeta(\tau)=\dfrac{\tau^\alpha}{\xi(\alpha)}$ *in* (21) *we get,*

$$\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{2(\delta-\gamma)^\alpha}\int_\gamma^\delta\Big((\delta-\tau)^{\alpha-1}+(\tau-\gamma)^{\alpha-1}\Big)\mathcal{S}_p(\tau,\cdot)d\tau=\frac{\delta-\gamma}{2}\sum_{k=1}^{4}\mathcal{I}_k,$$

where

$$\mathcal{I}_1=\int_0^{\frac{1}{2}}\tau^\alpha\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,\qquad \mathcal{I}_2=\int_0^{\frac{1}{2}}(-\tau^\alpha)\,\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)d\tau,$$
$$\mathcal{I}_3=\int_{\frac{1}{2}}^1(\tau^\alpha-1)\,\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,\qquad \mathcal{I}_4=\int_{\frac{1}{2}}^1(1-\tau^\alpha)\,\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)d\tau,$$

referred to [6].

- *For* $\zeta(\tau)=\dfrac{\tau^{\frac{\alpha}{k}}}{k\xi_k(\alpha)}$*, we have,*

$$\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{2k(\delta-\gamma)^{\frac{\alpha}{k}}}\int_\gamma^\delta\Big((\delta-\tau)^{\frac{\alpha}{k}-1}+(\tau-\gamma)^{\frac{\alpha}{k}-1}\Big)\mathcal{S}_p(\tau,\cdot)d\tau=\frac{\delta-\gamma}{2}\sum_{k=1}^{4}\mathcal{I}_k,$$

where

$$\mathcal{I}_1=\int_0^{\frac{1}{2}}\tau^{\frac{\alpha}{k}}\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,$$
$$\mathcal{I}_2=\int_0^{\frac{1}{2}}\left(-\tau^{\frac{\alpha}{k}}\right)\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)d\tau,$$
$$\mathcal{I}_3=\int_{\frac{1}{2}}^1\left(\tau^{\frac{\alpha}{k}}-1\right)\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,$$
$$\mathcal{I}_4=\int_{\frac{1}{2}}^1\left(1-\tau^{\frac{\alpha}{k}}\right)\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)d\tau.$$

- *For* $\zeta(\tau) = \tau(\delta-\tau)^{\alpha-1}$ *with* $\mathcal{S}_p$ *being symmetric to* $\frac{(\gamma+\delta)}{2}$, *the result becomes,*

$$\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{\alpha}{2(\delta^\alpha-\gamma^\alpha)}\int_\gamma^\delta \tau^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau = \frac{\alpha(\delta-\gamma)}{2\left(\delta^\alpha-\gamma^\alpha\right)}\sum_{k=1}^{4}\mathcal{J}k,$$

where

$$\begin{aligned}
\mathcal{J}1 &= \frac{1}{\alpha}\int_0^{\frac{1}{2}}\left(\delta^\alpha - [\delta-(\delta-\gamma)\tau]^\alpha\right)\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,\\
\mathcal{J}2 &= -\frac{1}{\alpha}\int_0^{\frac{1}{2}}\left(\delta^\alpha - [\delta-(\delta-\gamma)\tau]^\alpha\right)\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)d\tau,\\
\mathcal{J}3 &= -\frac{1}{\alpha}\int_{\frac{1}{2}}^{1}\left([\delta-(\delta-\gamma)\tau]^\alpha - \gamma^\alpha\right)\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,\\
\mathcal{J}4 &= \frac{1}{\alpha}\int_{\frac{1}{2}}^{1}\left([\delta-(\delta-\gamma)\tau]^\alpha - \gamma^\alpha\right)\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)d\tau.
\end{aligned}$$

- *For* $\zeta(\tau) = \frac{\tau}{\alpha}\exp\left(-\frac{(1-\alpha)}{\alpha}\tau\right)$, *where* $\alpha \in [0,1]$, *the following is true,*

$$\mathcal{S}_p(\gamma,\cdot)+\mathcal{S}_p(\delta,\cdot) - \frac{\alpha-1}{2(1-\exp(\mathcal{A}))}\left[{}_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot) +{}_{\delta^-}\ I_\zeta\mathcal{S}_p(\gamma,\cdot)\right] = \frac{(\delta-\gamma)(\alpha-1)}{2(\exp(\mathcal{A})-1)}\sum_{k=1}^{4}\mathcal{J}k,$$

where $\mathcal{A} = \frac{\alpha-1}{\alpha}(\delta-\gamma)$ *and*

$$\begin{aligned}
\mathcal{J}1 &= \int_0^{\frac{1}{2}}(\exp(\mathcal{A}\tau)-1)\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,\\
\mathcal{J}2 &= \int_0^{\frac{1}{2}}(1-\exp(\mathcal{A}\tau))\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)d\tau,\\
\mathcal{J}3 &= \int_{\frac{1}{2}}^{1}(\exp(\mathcal{A}\tau)-\exp(\mathcal{A}))\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau,\\
\mathcal{J}4 &= \int_{\frac{1}{2}}^{1}(\exp(\mathcal{A})-\exp(\mathcal{A}\tau))\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)d\tau.
\end{aligned}$$

In the next two theorems, we expand the estimations concerning the left-hand side of a Hermite-Hadamard type inequality for stochastic processes, specifically those with first derivative absolute values exhibiting P-convexity and Quasi-convexity.

Theorem 2.1. *Consider an MS-D stochastic process* $\mathcal{S}_p : \mathcal{I}\times\mathcal{E}\to\mathbb{R}$, *and* γ *and* δ *are elements of* $\mathcal{I}^\circ$, *with* $\gamma\le\delta$. *If* $\left|\mathcal{S}_p'\right|$ *is P-convex on the interval* $[\gamma,\delta]$, *the following inequality for generalized fractional integrals holds,*

$$\begin{aligned}
&\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{1}{2\Psi(1)}\left[{}_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+{}_{\delta^-}\ I_\zeta\mathcal{S}_p(\gamma,\cdot)\right]\right|\\
&\qquad\le \frac{(\delta-\gamma)}{\Psi(1)}\left[\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\right]\left(\int_0^{\frac{1}{2}}|\Psi(\tau)|d\tau + \int_{\frac{1}{2}}^{1}|\Phi(\tau)|d\tau\right).
\end{aligned} \tag{24}$$

Proof. By using Lemma 2.1 we get,

$$\begin{aligned}
&\left|\mathcal{S}_p\left(\frac{(\gamma+\delta)}{2},\cdot\right)-\frac{1}{2\Psi(1)}\left[{}_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+{}_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\right]\right|\\
&\leq\frac{(\delta-\gamma)}{2\Psi(1)}\left|\int_0^{\frac{1}{2}}\Psi(\tau)\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau+\int_0^{\frac{1}{2}}(-\Psi(\tau))\mathcal{S}_p'\left(\tau\gamma+(1-\tau)\delta,\cdot\right)d\tau\right|\\
&\quad+\frac{(\delta-\gamma)}{2\Psi(1)}\left|\int_0^{\frac{1}{2}}\Phi(\tau)\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)d\tau+\int_0^{\frac{1}{2}}(-\Phi(\tau))\mathcal{S}_p'\left(\tau\gamma+(1-\tau)\delta,\cdot\right)d\tau\right|\\
&\leq\frac{(\delta-\gamma)}{2\Psi(1)}\int_0^{\frac{1}{2}}(|\Psi(\tau)|+|-\Psi(\tau)|)d\tau\left(\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\right)\\
&\quad+\frac{(\delta-\gamma)}{2\Psi(1)}\int_{\frac{1}{2}}^{1}(|-\Phi(\tau)|+|\Phi(\tau)|)d\tau\left(\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\right).
\end{aligned}$$

Using the P-convexity of $|\mathcal{S}_p'|$. ☐

Remark 2.2.

- *For $\zeta(\tau)=\tau$, we obtain a result akin to that discovered in [6],*

$$\begin{aligned}
&\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{1}{(\delta-\gamma)}\int_\gamma^\delta\mathcal{S}_p(\tau,\cdot)d\tau\right|\\
&\leq\frac{(\delta-\gamma)}{(\delta-\gamma)}\Big[\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\Big]\int_0^{\frac{1}{2}}|(\delta-\gamma)\tau|d\tau+\int_{\frac{1}{2}}^{1}|(\delta-\gamma)(1-\tau)|d\tau\\
&\leq(\delta-\gamma)\Big[\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\Big]\left(\int_0^{\frac{1}{2}}\tau d\tau+\int_{\frac{1}{2}}^{1}(1-\tau)d\tau\right)\\
&\leq\frac{(\delta-\gamma)}{4}\Big(\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\Big).
\end{aligned}$$

- *For $\zeta(\tau)=\frac{\tau^\alpha}{\xi(\alpha)}$, we get a reseult similar to that found in [3],*

$$\begin{aligned}
&\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{2(\delta-\gamma)^\alpha}\int_\gamma^\delta\Big((\delta-\tau)^{\alpha-1}+(\tau-\gamma)^{\alpha-1}\Big)\mathcal{S}_p(\tau,\cdot)d\tau\right|\\
&\leq(\delta-\gamma)\Big(\,|\mathcal{S}_p'(\gamma,\cdot)+|\,\mathcal{S}_p'(\delta,\cdot)\,|\,\Big)\left(\int_0^{\frac{1}{2}}\tau^\alpha d\tau+\int_{\frac{1}{2}}^{2}\left(1-\tau^\alpha\right)d\tau\right)\\
&\leq\frac{(\delta-\gamma)}{2(\alpha+1)}\left(\frac{1}{2^{\alpha-1}}+(\alpha-1)\right)\Big[\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\Big].
\end{aligned}$$

- *For $\zeta(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\xi_k(\alpha)}$, we have,*

$$\begin{aligned}
&\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{2k(\delta-\gamma)^{\frac{\alpha}{k}}}\int_\gamma^\delta\Big((\delta-\tau)^{\frac{\alpha}{k}-1}+(\tau-\gamma)^{\frac{\alpha}{k}-1}\Big)\mathcal{S}_p(\tau,\cdot)d\tau\right|\\
&\leq(\delta-\gamma)\Big(\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\Big)\left(\int_0^{\frac{1}{2}}\left|\tau^{\frac{\alpha}{k}}\right|d\tau+\int_0^{1}\left|1-\tau^{\frac{\alpha}{k}}\right|d\tau\right)\\
&\leq(\delta-\gamma)\Big[\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\Big]\left(\frac{1}{2}+\frac{1}{\frac{\alpha}{k}+1}\left(\frac{1}{2^{\frac{\alpha}{k}}}-1\right)\right).
\end{aligned}$$

Corollary 2.1. *Assuming that Theorem 2.1 is verified, with $\zeta(\tau)=\tau(\delta-\tau)^{\alpha-1}$ and $\mathcal{S}_p$ being symetric to $\frac{(\gamma+\delta)}{2}$, the result becomes,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{\delta^\alpha-\gamma^\alpha}\int_\gamma^\delta \tau^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau\right| \\ \leq \frac{1}{(\delta^\alpha-\gamma^\alpha)}\left[\left(\gamma^{\alpha+1}+\delta^{\alpha+1}\right)\frac{\alpha}{\alpha+1}-\left(\gamma^\alpha+\delta^\alpha\right)\frac{\gamma+\delta}{2}+\frac{(\gamma+\delta)^{\alpha+1}}{2^\alpha(\alpha+1)}\right]\times\left[\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\right]. \tag{25}$$

Proof. For $\zeta(\tau)=\tau(\delta-\tau)^{\alpha-1}$ we have,

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{\delta^\alpha-\gamma^\alpha}\int_\gamma^\delta \tau^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau\right| \\ \leq \frac{\alpha(\delta-\gamma)}{\delta^\alpha-\gamma^\alpha}\left[\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\right]\left(\int_0^{\frac12}|\Psi(\tau)|d\tau+\int_{\frac12}^1|\Phi(\tau)|d\tau\right).$$

We compute the integrals as follows,

$$\int_0^{\frac12}|\Psi(\tau)|d\tau=\frac{1}{\alpha}\int_0^{\frac12}\left|\delta^\alpha-[\delta-(\delta-\gamma)\tau]^\alpha\right|d\tau \\ =\frac{1}{\alpha(\delta-\gamma)}\int_{\gamma+\delta}^{\delta}\left|\delta^\alpha-s^\alpha\right|ds=\frac{1}{\alpha(\delta-\gamma)}\left[\frac{\alpha}{\alpha+1}\delta^{\alpha+1}+\frac{(\gamma+\delta)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}-\delta^\alpha\frac{\gamma+\delta}{2}\right],$$

and

$$\int_{\frac12}^1|\Phi(\tau)|d\tau=\frac{1}{\alpha}\int_{\frac12}^1\left|[\delta-(\delta-\gamma)\tau]^\alpha-\gamma^\alpha\right|d\tau \\ =\frac{1}{\alpha(\delta-\gamma)}\int_\gamma^{\frac{\gamma+\delta}{2}}\left|s^\alpha-\gamma^\alpha\right|ds=\frac{1}{\alpha(\delta-\gamma)}\left[\frac{\alpha}{\alpha+1}\gamma^{\alpha+1}+\frac{(\gamma+\delta)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}-\gamma^\alpha\frac{\gamma+\delta}{2}\right].$$

Thus, we obtain the result. □

Corollary 2.2. *In the context of Theorem 2.1, for $\zeta(\tau)=\frac{\tau}{\alpha}\exp\left(-\frac{(1-\alpha)}{\alpha}\tau\right)$, $\alpha\in[0,1]$ the result is,*

$$\left|\mathcal{S}_p(\gamma,\cdot)+\mathcal{S}_p(\delta,\cdot)-\frac{(\alpha-1)}{2(1-\exp(\mathcal{A}))}\left[{}_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+{}_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\right]\right| \\ \leq\frac{(\delta-\gamma)}{\alpha(1-\exp(\mathcal{A}))}\left(\frac12(1-\exp(\mathcal{A}))+\frac{1}{\mathcal{A}}\left(1-2\exp\left(\frac{\mathcal{A}}{2}\right)+\exp(\mathcal{A})\right)\right)\times\left[\left|\mathcal{S}_p'(\gamma,\cdot)\right|+\left|\mathcal{S}_p'(\delta,\cdot)\right|\right],$$

where $\mathcal{A}=\frac{\alpha-1}{\alpha}(\delta-\gamma)$.

Proof. By taking $\zeta(\tau) = \frac{\tau}{\alpha} \exp\left(-\frac{(1-\alpha)}{\alpha}\tau\right)$, $\alpha \in [0,1]$, we get,

$$\left|\mathcal{S}_p(\gamma,\cdot) + \mathcal{S}_p(\delta,\cdot) - \frac{\alpha-1}{2(1-\exp(\mathcal{A}))}\left[{}_{\gamma^+}I_\zeta \mathcal{S}_p(\delta,\cdot) + {}_{\delta-}\, I_\zeta \mathcal{S}_p(\gamma,\cdot)\right]\right|$$
$$\leq \frac{\delta-\gamma}{\alpha(1-\exp(\mathcal{A}))}\left(\int_0^{\frac{1}{2}} |\exp(\mathcal{A}\tau) - 1| d\tau + \int_{\frac{1}{2}}^1 |\exp(\mathcal{A}) - \exp(\mathcal{A}\tau)| d\tau\right) \times \left[\left|\mathcal{S}_p'(\gamma,\cdot)\right| + \left|\mathcal{S}_p'(\delta,\cdot)\right|\right]$$
$$= \frac{\delta-\gamma}{\alpha(1-\exp(A))}\left(\frac{1}{2}(1-\exp(\mathcal{A})) + \frac{1}{\mathcal{A}}\left(1 - 2\exp\left(\frac{\mathcal{A}}{2}\right) + \exp(\mathcal{A})\right)\right) \times \left[\left|\mathcal{S}_p'(\gamma,\cdot)\right| + \left|\mathcal{S}_p'(\delta,\cdot)\right|\right].$$

□

Theorem 2.2. *Consider an MS-D stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$, and γ and δ are elements of $\mathcal{I}^\circ$, with $\gamma \leq \delta$. If $\left|\mathcal{S}_p'\right|$ is quasi-convex on the interval $[\gamma,\delta]$, the following inequality for generalized fractional integrals holds,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{1}{2\Psi(1)}\left[{}_{\gamma^+}I_\zeta \mathcal{S}_p(\delta,\cdot) + {}_{\delta-}\, I_\zeta \mathcal{S}_p(\gamma,\cdot)\right]\right|$$
$$\leq \frac{(\delta-\gamma)}{\Psi(1)} \operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|, \left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\int_0^{\frac{1}{2}} |\Psi(\tau)|\, d\tau + \int_{\frac{1}{2}}^1 |\Phi(\tau)|\, d\tau\right). \tag{26}$$

Proof. By using Lemma 2.1 and by using the Quasi-convexity of $\left|\mathcal{S}_p'\right|$ we get the result. □

Remark 2.3.

- *For $\zeta(\tau) = \tau$, we obtain a result comparable to that discovered in [6],*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{1}{(\delta-\gamma)}\int_\gamma^\delta \mathcal{S}_p(\tau,\cdot) d\tau\right|$$
$$\leq \frac{(\delta-\gamma)}{(\delta-\gamma)} \operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|, \left|\mathcal{S}_p'(\delta,\cdot)\right|\right\} \int_0^{\frac{1}{2}} |(\delta-\gamma)\tau|\, d\tau + \int_{\frac{1}{2}}^1 |(\delta-\gamma)(1-\tau)|\, dt$$
$$\leq (\delta-\gamma) \operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|, \left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\int_0^{\frac{1}{2}} \tau d\tau + \int_{\frac{1}{2}}^1 (1-\tau) d\tau\right)$$
$$\leq \frac{(\delta-\gamma)}{4} \operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|, \left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}.$$

- *For $\zeta(\tau) = \frac{\tau^\alpha}{\xi(\alpha)}$, we get a result similar to the one found in [3],*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{\alpha}{2(\delta-\gamma)^\alpha}\int_\gamma^\delta \left((\delta-\tau)^{\alpha-1} + (\tau-\gamma)^{\alpha-1}\right)\mathcal{S}_p(\tau,\cdot) d\tau\right|$$
$$\leq (\delta-\gamma) \operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|, \left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\int_0^{\frac{1}{2}} \tau^\alpha d\tau + \int_{\frac{1}{2}}^2 (1-\tau^\alpha)\, d\tau\right)$$
$$\leq \frac{(\delta-\gamma)}{2(\alpha+1)}\left(\frac{1}{2^{\alpha-1}} + (\alpha-1)\right) \operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|, \left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}.$$

- *For* $\zeta(\tau)=\dfrac{\tau^{\frac{\alpha}{k}}}{k\xi_k(\alpha)}$, *we have,*

$$\left|\mathcal{S}_p\left(\frac{(\gamma+\delta)}{2},\cdot\right)-\frac{\alpha}{2k(\delta-\gamma)^{\frac{\alpha}{k}}}\int_\gamma^\delta\Big((\delta-\tau)^{\frac{\alpha}{k}-1}+(\tau-\gamma)^{\frac{\alpha}{k}-1}\Big)\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq(\delta-\gamma)\operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\int_0^{\frac{1}{2}}\left|\tau^{\frac{\alpha}{k}}\right|d\tau+\int_0^1\left|1-\tau^{\frac{\alpha}{k}}\right|d\tau\right)$$
$$\leq(\delta-\gamma)\operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\frac{1}{2}+\frac{1}{\frac{\alpha}{k}+1}\left(\frac{1}{2^{\frac{\alpha}{k}}}-1\right)\right).$$

Corollary 2.3. *Assuming Theorem 2.2 is verified for* $\zeta(\tau)=\tau(\delta-\tau)^{\alpha-1}$ *and* $\mathcal{S}_p$ *being symmetric to* $\dfrac{(\gamma+\delta)}{2}$, *we get,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{\delta^\alpha-\gamma^\alpha}\int_\gamma^\delta\tau^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq\frac{1}{(\delta^\alpha-\gamma^\alpha)}\operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}$$
$$\times\left[\left(\gamma^{\alpha+1}+\delta^{\alpha+1}\right)\frac{\alpha}{\alpha+1}-(\gamma^\alpha+\delta^\alpha)\frac{\gamma+\delta}{2}+\frac{(\gamma+\delta)^{\alpha+1}}{2^\alpha(\alpha+1)}\right].$$

Proof. For $\zeta(\tau)=\tau(\delta-\tau)^{\alpha-1}$, we obtain,

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{\delta^\alpha-\gamma^\alpha}\int_\gamma^\delta\tau^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq\frac{\alpha(\delta-\gamma)}{\delta^\alpha-\gamma^\alpha}\operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\int_0^{\frac{1}{2}}|\Psi(\tau)|d\tau+\int_{\frac{1}{2}}^1|\Phi(\tau)|d\tau\right).$$

By computing two integrals, we get,

$$\int_0^{\frac{1}{2}}|\Psi(\tau)|d\tau=\frac{1}{\alpha}\int_0^{\frac{1}{2}}\left|\delta^\alpha-[\delta-(\delta-\gamma)\tau]^\alpha\right|d\tau,$$

and

$$\int_{\frac{1}{2}}^1|\Phi(\tau)|d\tau=\frac{1}{\alpha}\int_{\frac{1}{2}}^1\left|[\delta-(\delta-\gamma)\tau]^\alpha-\gamma^\alpha\right|dt,$$

which gives us the result. ∎

Corollary 2.4. *Under the assumption of* $Theorem$ *2.2 with* $\zeta(\tau)=\dfrac{\tau}{\alpha}\exp\left(-\dfrac{(1-\alpha)}{\alpha}\tau\right)$ $\alpha\in[0,1]$, *then we have,*

$$\left|\mathcal{S}_p(\gamma,\cdot)+\mathcal{S}_p(\delta,\cdot)-\frac{\alpha-1}{2(1-\exp(\mathcal{A}))}\Big[{}_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+{}_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\Big]\right|$$
$$\leq\frac{\delta-\gamma}{\alpha(1-\exp(\mathcal{A}))}\operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\frac{1}{2}(1-\exp(\mathcal{A}))+\frac{1}{\mathcal{A}}\left(1-2\exp\left(\frac{\mathcal{A}}{2}\right)+\exp(\mathcal{A})\right)\right),$$

where $\mathcal{A}=\dfrac{\alpha-1}{\alpha}(\delta-\gamma)$.

Proof. We take $\zeta(\tau) = \frac{\tau}{\alpha}\exp\left(-\frac{(1-\alpha)}{\alpha}\tau\right)$, $\alpha \in [0,1]$, we have,

$$\left|\mathcal{S}_p(\gamma,\cdot)+\mathcal{S}_p(\delta,\cdot)-\frac{\alpha-1}{2(1-\exp(\mathcal{A}))}\left[_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\right]\right|$$
$$\leq \frac{(\delta-\gamma)}{\alpha(1-\exp(A))}\operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\int_0^{\frac{1}{2}}|\exp(\mathcal{A}\tau)-1|d\tau+\int_{\frac{1}{2}}^1|\exp(\mathcal{A})-\exp(\mathcal{A}\tau)|d\tau\right)$$
$$=\frac{(\delta-\gamma)}{\alpha(1-\exp(A))}\operatorname{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\frac{1}{2}(1-\exp(\mathcal{A}))+\frac{1}{\mathcal{A}}\left(1-2\exp\left(\frac{\mathcal{A}}{2}\right)+\exp(\mathcal{A})\right)\right).$$

□

Theorem 2.3. *Consider an MS-D stochastic process $\mathcal{S}_p : \mathcal{I}\times\mathcal{E}\to\mathbb{R}$, and γ and δ are elements of $\mathcal{I}^\circ$, with $\gamma\leq\delta$. Assuming that $\left|\mathcal{S}_p'\right|^\sigma$ is P- convex on $[\gamma,\delta]$, for some $\sigma>1$, then the following inequality holds,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{1}{2\Psi(1)}\left[_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\right]\right|$$
$$\leq\frac{(\delta-\gamma)}{\Psi(1)}\left[\left(\int_0^{\frac{1}{2}}|\Psi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}+\left(\int_{\frac{1}{2}}^1|\Phi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}\right]\left(\left|\mathcal{S}_p'(\delta,\cdot)\right|^\sigma+\left|\mathcal{S}_p'(\gamma,\cdot)\right|^\sigma\right)^{\frac{1}{\sigma}}, \tag{27}$$

where $\frac{1}{\rho}+\frac{1}{\sigma}=1$.

Proof. With Lemma 2.1, we get,

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{1}{2\Psi(1)}\left[_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\right]\right|$$
$$\leq\frac{(\delta-\gamma)}{2\Psi(1)}\left[\int_0^{\frac{1}{2}}|\Psi(\tau)|\left|\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)\right|d\tau+\int_0^{\frac{1}{2}}|-\Psi(\tau)|\left|\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)\right|d\tau\right]$$
$$+\frac{(\delta-\gamma)}{2\Psi(1)}\left[\int_{\frac{1}{2}}^1|-\Phi(\tau)|\left|\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)\right|d\tau+\int_{\frac{1}{2}}^1|\Phi(\tau)|\left|\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot))\right|d\tau\right].$$

And by applying the Hölder inequality, we obtain,

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{1}{2\Psi(1)}\left[_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\right]\right|$$
$$\leq\frac{(\delta-\gamma)}{2\Psi(1)}\left[\left(\int_0^{\frac{1}{2}}|\Psi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)\right|^\sigma\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_0^{\frac{1}{2}}|\Psi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)\right|^\sigma\right)^{\frac{1}{\sigma}}\right]$$
$$+\frac{(\delta-\gamma)}{2\Psi(1)}\left[\left(\int_{\frac{1}{2}}^1|\Phi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{S}_p'(\tau\delta+(1-\tau)\gamma,\cdot)\right|^\sigma\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1|\Phi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}\left(\int_0^{\frac{1}{2}}\left|\mathcal{S}_p'(\tau\gamma+(1-\tau)\delta,\cdot)\right|^\sigma\right)^{\frac{1}{\sigma}}\right].$$

Finally by considering the P-convexity of $\left|\mathcal{S}_p'\right|^\sigma$ we get,

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{1}{2\Psi(1)}\left[_{\gamma^+}I_\zeta\mathcal{S}_p(\delta,\cdot)+_{\delta^-}I_\zeta\mathcal{S}_p(\gamma,\cdot)\right]\right|$$
$$\leq\frac{(\delta-\gamma)}{\Psi(1)}\left[\left(\int_0^{\frac{1}{2}}|\Psi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}+\left(\int_{\frac{1}{2}}^1|\Phi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}\right]\left(\left|\mathcal{S}_p'(\delta,\cdot)\right|^\sigma+\left|\mathcal{S}_p'(\gamma,\cdot)\right|^\sigma\right)^{\frac{1}{\sigma}}.$$

□

Remark 2.4.

- *For $\zeta(\tau)=\tau$, we get*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{1}{(\delta-\gamma)}\int_\gamma^\delta\mathcal{S}_p(\tau,\cdot)d\tau\right|\leq\frac{2(\delta-\gamma)}{(p+1)^{\frac{1}{\rho}}}\left(\frac{1}{2^{\rho+1}}\right)^{\frac{1}{\rho}}\left[\left|\mathcal{S}_p'(\gamma,\cdot)\right|^\sigma+\left|\mathcal{S}_p'(\delta,\cdot)\right|^\sigma\right]^{\frac{1}{\sigma}}.$$

- *For $\zeta(\tau)=\dfrac{\tau^\alpha}{\xi(\alpha)}$, we have*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{2(\delta-\gamma)^\alpha}\int_\gamma^\delta\left((\delta-\tau)^{\alpha-1}+(\tau-\gamma)^{\alpha-1}\right)\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq(\delta-\gamma)\left(\left(\frac{1}{(\alpha\rho+1)2^{\alpha\rho+1}}\right)^{\frac{1}{\rho}}+\left(\frac{1}{2}+\frac{1}{\alpha\rho+1}\left(\frac{1}{2^{\alpha\rho+1}}-1\right)\right)^{\frac{1}{\rho}}\right)$$
$$\times\left[\left|\mathcal{S}_p'(\gamma,\cdot)\right|^\sigma+\left|\mathcal{S}_p'(\delta,\cdot)\right|^\sigma\right]^{\frac{1}{\sigma}}.$$

- *For $\zeta(\tau)=\dfrac{\tau^{\frac{\alpha}{k}}}{k\xi_k(\alpha)}$, we obtain*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{2k(\delta-\gamma)^{\frac{\alpha}{k}}}\int_\gamma^\delta\left((\delta-\tau)^{\frac{\alpha}{k}-1}+(\tau-\gamma)^{\frac{\alpha}{k}-1}\right)\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq\frac{2(\delta-\gamma)}{\left(\frac{\alpha\rho}{k}+1\right)^{\frac{1}{\rho}}\left(2^{\frac{\alpha\rho}{k}+1}\right)^{\frac{1}{\rho}}}\left[\left|\mathcal{S}_p'(\gamma,\cdot)\right|^\sigma+\left|\mathcal{S}_p'(\delta,\cdot)\right|^\sigma\right]^{\frac{1}{\sigma}}.$$

Corollary 2.5. *Under the assumption of Theorem 2.3 with $\zeta(\tau)=\tau(\delta-\tau)^{\alpha-1}$ and $\mathcal{S}_p$ is a symmetric to $\dfrac{(\gamma+\delta)}{2}$, then we have,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right)-\frac{\alpha}{\delta^\alpha-\gamma^\alpha}\int_\gamma^\delta\tau^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq\frac{(\delta-\gamma)^{\frac{1}{\sigma}}}{(\delta^\alpha-\gamma^\alpha)}\left[\left|\mathcal{S}_p'(\gamma,\cdot)\right|^\sigma+\left|\mathcal{S}_p'(\delta,\cdot)\right|^\sigma\right]^{\frac{1}{\sigma}}\left(\left[\left(\frac{\alpha\rho}{\alpha\rho+1}-\frac{\gamma+\delta}{2}\right)\delta^{\alpha\rho+1}+\frac{(\gamma+\delta)^{\alpha\rho+1}}{2^{\alpha\rho+1}(\alpha\rho+1)}\right]^{\frac{1}{\rho}}\right.$$
$$\left.+\left[\left(\frac{\alpha\rho}{\alpha\rho+1}-\frac{\gamma+\delta}{2}\right)\gamma^{\alpha\rho+1}+\frac{(\gamma+\delta)^{\alpha\rho+1}}{2^{\alpha\rho+1}(\alpha\rho+1)}\right]^{\frac{1}{\rho}}\right).$$

Proof. We calculate the following integrals, using the inequality $\mathcal{A} \geq \mathcal{B} > 0$ and $(\mathcal{A}-\mathcal{B})^\sigma \leq \mathcal{A}^\sigma - \mathcal{B}^\sigma$ for $\sigma > 1$.

$$\begin{aligned}\int_0^{\frac{1}{2}} |\Psi(\tau)|^\rho \, d\tau &= \frac{1}{\alpha\gamma^\rho} \int_0^{\frac{1}{2}} |\delta^\alpha - [\delta - (\delta-\gamma)\tau]^\alpha| \, \gamma^\alpha \, d\tau \\ &\leq \frac{1}{\alpha\gamma^\rho(\delta-\gamma)} \int_{\frac{\gamma+\delta}{2}}^{\delta} \left(\delta^{\alpha\rho} - s^{\alpha\rho}\right) ds \\ &\leq \frac{1}{\alpha\gamma^\rho(\delta-\gamma)} \left[\left(\frac{\alpha\rho}{\alpha\rho+1}\delta^{\alpha\rho+1} - \frac{(\gamma+\delta)^{\alpha\rho+1}}{2^{\alpha\rho+1}(\alpha\rho+1)}\right)\right],\end{aligned}$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^1 |\Phi(\tau)|^\rho \, d\tau &= \frac{1}{\alpha\gamma^\rho} \int_{\frac{1}{2}}^1 |[\delta - (\delta-\gamma)\tau]^\alpha - \gamma^\alpha|^\rho \, d\tau \\ &\leq \frac{1}{\alpha\gamma^\rho(\delta-\gamma)} \int_\gamma^{\frac{\gamma+\delta}{2}} \left(s^{\alpha\rho} - \gamma^{\alpha\rho}\right) ds \\ &= \frac{1}{\alpha\gamma^\rho(\delta-\gamma)} \left[\left(\frac{\alpha\rho}{\alpha\rho+1} s^{\alpha\rho+1}\right)_\gamma^{\frac{\gamma+\delta}{2}}\right] \\ &= \frac{1}{\alpha\gamma^\rho(\delta-\gamma)} \left[\left(\frac{(\gamma+\delta)^{\alpha\rho+1}}{2^{\alpha\rho+1}(\alpha\rho+1)} - \frac{\gamma^{\alpha\rho+1}}{\alpha\rho+1}\right)\right],\end{aligned}$$

which gives us the result. □

Theorem 2.4. *Consider an MS-D stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$, and γ and δ are elements of $\mathcal{I}^\circ$, with $\gamma \leq \delta$. If we assume that $|\mathcal{S}_p'|^\sigma$ is Quasi- convex on $[\gamma, \delta]$, for some $\sigma > 1$, then the following inequality holds,*

$$\begin{aligned}&\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2}, \cdot\right) - \frac{1}{2\Psi(1)}\left[{}_{\gamma^+}I_\zeta \mathcal{S}_p(\delta, \cdot) + {}_{\delta^-}\, I_\zeta \mathcal{S}_p(\gamma, \cdot)\right]\right| \\ &\quad \leq \frac{(\delta-\gamma)}{\Psi(1)}\left[\left(\int_0^{\frac{1}{2}} |\Psi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}} + \left(\int_{\frac{1}{2}}^1 |\Phi(\tau)|^\rho d\tau\right)^{\frac{1}{\rho}}\right] \text{Max}\left\{ \left|\mathcal{S}_p'(\gamma, \cdot)\right|, \left|\mathcal{S}_p'(\delta, \cdot)\right| \right\},\end{aligned} \tag{28}$$

where $\dfrac{1}{\rho} + \dfrac{1}{\sigma} = 1$.

Proof. We obtain the result by considering the Hölder inequality and the quasi-convexity of $|\mathcal{S}_p'|^\sigma$ on Lemma 2.1. □

Remark 2.5.

- *For $\zeta(\tau) = \tau$, we get,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2}, \cdot\right) - \frac{1}{(\delta-\gamma)}\int_\gamma^\delta \mathcal{S}_p(\tau, \cdot) d\tau\right| \leq \frac{2(\delta-\gamma)}{(p+1)^{\frac{1}{\rho}}}\left(\frac{1}{2^{\rho+1}}\right)^{\frac{1}{\rho}} \text{Max}\left\{ \left|\mathcal{S}_p'(\gamma, \cdot)\right|, \left|\mathcal{S}_p'(\delta, \cdot)\right| \right\}.$$

- *For* $\zeta(\tau) = \dfrac{\tau^{\alpha}}{\xi(\alpha)}$, *we have,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{\alpha}{2(\delta-\gamma)^{\alpha}}\int_{\gamma}^{\delta}\Big((\delta-\tau)^{\alpha-1}+(\tau-\gamma)^{\alpha-1}\Big)\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq (\delta-\gamma)\,\mathrm{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\left(\frac{1}{(\alpha\rho+1)2^{\alpha\rho+1}}\right)^{\frac{1}{\rho}}\right.$$
$$\left.+\left(\frac{1}{2}+\frac{1}{\alpha\rho+1}\left(\frac{1}{2^{\alpha\rho+1}}-1\right)\right)^{\frac{1}{\rho}}\right).$$

- *For* $\zeta(\tau) = \dfrac{\tau^{\frac{\alpha}{k}}}{k\xi_k(\alpha)}$, *we obtain,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{\alpha}{2k(\delta-\gamma)^{\frac{\alpha}{k}}}\int_{\gamma}^{\delta}\Big((\delta-\tau)^{\frac{\alpha}{k}-1}+(\tau-\gamma)^{\frac{\alpha}{k}-1}\Big)\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq \frac{2(\delta-\gamma)}{\left(\frac{\alpha\rho}{k}+1\right)^{\frac{1}{\rho}}\left(2^{\frac{\alpha\rho}{k}+1}\right)^{\frac{1}{\rho}}}\,\mathrm{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}.$$

Corollary 2.6. *Assuming Theorem 2.4 is satisfied for* $\zeta(\tau) = \tau(\delta-\tau)^{\alpha-1}$ *and* $\mathcal{S}_p$ *being symmetric to* $\dfrac{(\gamma+\delta)}{2}$, *we obtain,*

$$\left|\mathcal{S}_p\left(\frac{\gamma+\delta}{2},\cdot\right) - \frac{\alpha}{\delta^{\alpha}-\gamma^{\alpha}}\int_{\gamma}^{\delta}\tau^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau\right|$$
$$\leq \frac{(\delta-\gamma)^{\frac{1}{\sigma}}}{(\delta^{\alpha}-\gamma^{\alpha})}\,\mathrm{Max}\left\{\left|\mathcal{S}_p'(\gamma,\cdot)\right|,\left|\mathcal{S}_p'(\delta,\cdot)\right|\right\}\left(\left[\left(\frac{\alpha\rho}{\alpha\rho+1}-\frac{\gamma+\delta}{2}\right)\delta^{\alpha\rho+1}+\frac{(\gamma+\delta)^{\alpha\rho+1}}{2^{\alpha\rho+1}(\alpha\rho+1)}\right]^{\frac{1}{\rho}}\right.$$
$$\left.+\left[\left(\frac{\alpha\rho}{\alpha\rho+1}-\frac{\gamma+\delta}{2}\right)\gamma^{\alpha\rho+1}+\frac{(\gamma+\delta)^{\alpha\rho+1}}{2^{\alpha\rho+1}(\alpha\rho+1)}\right]^{\frac{1}{\rho}}\right).$$

Proof. We calculate the following integrals, using the inequality $\mathcal{A} \geq \mathcal{B} > 0$ and $(\mathcal{A}-\mathcal{B})^{\sigma} \leq \mathcal{A}^{\sigma}-\mathcal{B}^{\sigma}$ for $\sigma > 1$.

$$\int_0^{\frac{1}{2}}|\Psi(\tau)|^{\rho}\,d\tau = \frac{1}{\alpha\gamma^{\rho}}\int_0^{\frac{1}{2}}\left|\delta^{\alpha}-[\delta-(\delta-\gamma)\tau]^{\alpha}\right|^{\rho}\,d\tau$$
$$\leq \frac{1}{\alpha\gamma^{\rho}(\delta-\gamma)}\int_{\frac{\gamma+\delta}{2}}^{b}\left(\delta^{\alpha\rho}-s^{\alpha\rho}\right)\,ds$$
$$\leq \frac{1}{\alpha\gamma^{\rho}(\delta-\gamma)}\left[\left(\frac{\alpha\rho}{\alpha\rho+1}\delta^{\alpha\rho+1}-\frac{\gamma+\delta}{2}\right)+\frac{(\gamma+\delta)^{\alpha\rho+1}}{2^{\alpha\rho+1}(\alpha\rho+1)}\right],$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^{1} |\Phi(\tau)|^{\rho}\, d\tau &= \frac{1}{\alpha\gamma^{\rho}} \int_{\frac{1}{2}}^{1} \left|[\delta-(\delta-\gamma)\tau]^{\alpha}-\gamma^{\alpha}\right|^{\rho}\, d\tau \\ &\leq \frac{1}{\alpha\gamma^{\rho}(\delta-\gamma)} \int_{\gamma}^{\frac{\gamma+\delta}{2}} \left(s^{\alpha\rho}-\gamma^{\alpha\rho}\right) ds \\ &= \frac{1}{\alpha\gamma^{\rho}(\delta-\gamma)} \left[\left(\frac{\alpha\rho}{\alpha\rho+1} s^{\alpha\rho+1}\right)_{\gamma}^{\frac{\gamma+\delta}{2}}\right] \\ &= \frac{1}{\alpha\gamma^{\rho}(\delta-\gamma)} \left[\left(\frac{(\gamma+\delta)^{\alpha\rho+1}}{2^{\alpha\rho+1}(\alpha\rho+1)} - \frac{\gamma^{\alpha\rho+1}}{\alpha\rho+1}\right)\right].\end{aligned}$$

Thus, the result is complete. □

3 Conclusion

In this paper, a study was presented on the estimation of the left-hand side of a Hermite-Hadamard type inequality for stochastic processes. Specifically, the focus was on stochastic processes whose first derivatives had P-convex and quasi-convex absolute values. To achieve this, a generalized fractional integral was utilized. The approach allowed for the derivation of new and significant results concerning well-known fractional integral operators. The outcomes of this research provide valuable insights and could serve as a foundation for further investigations in this field. By examining and expanding upon these findings, researchers could uncover novel mathematical relationships and explore their applications in related areas. These contributions aim to enhance the understanding of the connections between convex functions, fractional integral operators, and stochastic processes.

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References

[1] B. Ahmad, A. Alsaedi, M. Kirane & B. T. Torebek (2019). Hermite–Hadamard, Hermite–Hadamard–Fejér, Dragomir–Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals. *Journal of Computational and Applied Mathematics*, *353*, 120–129. https://doi.org/10.1016/j.cam.2018.12.030.

[2] E. Halmaghi & P. Liljedahl (2015). Inequalities in the history of mathematics: From peculiarities to a hard discipline. *GeoGebra International Journal of Romania*, *4*, 43–56.

[3] M. Iqbal, M. Iqbal Bhatti & K. Nazeer (2015). Generalization of inequalities analogous to Hermite–Hadamard inequality via fractional integrals. *Bulletin of the Korean Mathematical Society*, *52*(3), 707–716. https://doi.org/10.4134/BKMS.2015.52.3.707.

[4] I. Karatzas & S. Shreve (1991). *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer Science & Business Media, New York. https://doi.org/10.1007/978-1-4612-0949-2.

[5] R. Khalil, M. Al Horani, A. Yousef & M. Sababheh (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, *264*, 65–70. https://doi.org/10.1016/j.cam.2014.01.002.

[6] U. S. Kirmaci (2004). Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Applied Mathematics and Computation*, *147*(1), 137–146. https://doi.org/10.1016/S0096-3003(02)00657-4.

[7] D. Kotrys (2012). Hermite–Hadamard inequality for convex stochastic processes. *Aequationes Mathematicae*, *83*, 143–151. https://doi.org/10.1007/s00010-011-0090-1.

[8] Y. Laarichi, M. Elkaf, A. Aloui & O. Rholam (2024). Transforming data with the arcsine distribution for random walks. *Mathematical Models in Engineering*, *10*(2), 98–108. https://doi.org/10.21595/mme.2024.24105.

[9] S. Mubeen & G. M. Habibullah (2012). k-fractional integrals and application. *International Journal of Contemporary Mathematical Sciences*, *7*(2), 89–94.

[10] M. E. Özdemir, M. Avcı & E. Set (2010). On some inequalities of Hermite–Hadamard type via m-convexity. *Applied Mathematics Letters*, *23*(9), 1065–1070. https://doi.org/10.1016/j.aml.2010.04.037.

[11] O. Rholam, M. Barmaki & D. Gretete (2023). Fractional integral inequalities of Hermite–Hadamard type for P-convex and Quasi-convex stochastic process. *Australian Journal of Mathematical Analysis and Applications*, *20*(1), Article ID: 10.

[12] M. Z. Sarikaya, H. Yildirim & Ö. Akin (2008). On generalized Riesz type potential with Lorentz distance. *Lobachevskii Journal of Mathematics*, *29*(1), 32–39. https://doi.org/10.1134/S1995080208010083.

[13] E. Set, M. Tomar & S. Maden (2014). Hermite–Hadamard type inequalities for $s-$convex stochastic processes in the second sense. *Turkish Journal of Analysis and Number Theory*, 2(6), 202–207. https://doi.org/10.12691/tjant-2-6-3.

[14] A. Skowroński (1995). On Wright-convex stochastic processes. *Annales Mathematicae Silesianae*, *9*, 29–32.

[15] D. Xiu (2010). *Numerical Methods for Stochastic Computations: A Spectral Method Approach*. Princeton University Press, Princeton, New Jersey. https://doi.org/10.2307/j.ctv7h0skv.